A Geometry of Nutrition[1]

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ABSTRACT The nutritional value of a given sample of food may be specified by a *nutrition holor* in a fictitious 3-space. This holor possesses *magnitude* and *character*. In most cases, we are interested primarily in character, and this can be represented by a point in a 2-space (the *nutrition triangle*). The triangle allows visualization of relations among foods and also the addition of nutritional values for a combination of foods. J. Nutr. *104:* 1535–1542, 1974.

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The worldwide importance of modern nutritional theory can hardly be overestimated. Nutrition calculations, however, have remained in the arithmetic stage and have not utilized the possibilities of algebraic and geometric formulation. In particular, a geometrization of nutrition allows one to *visualize* the relations among nutrients and thus to bypass much of the present routine computation, in accordance with the principle that a picture is worth a thousand words.

We wish to present, therefore, a possible geometrization of nutrition, based on the specification in terms of protein, lipid, and carbohydrate. Such a specification is admittedly rudimentary; but it has proved to be highly useful, and it can be supplemented by further requirements involving vitamins, minerals, amino acids, and fatty acids.

The nutrition triangle. Chemical analysis of food yields values of the mass (kg) of protein, lipid (fat), and carbohydrate. Instead of using these absolute values directly, however, it is usually more convenient to employ *relative values.* Expressed as fractions of the total mass of nutrients, we have [2]

n^1 = relative mass of protein;
n^2 = relative mass of lipid;
n^3 = relative mass of carbohydrate.

Because $n^1 + n^2 + n^3 = 1$, only two of these quantities are independent. Thus the *character* of a food may be specified by giving any two of the above coordinates. These coordinates can be plotted in a *nutrition triangle* (fig. 1).

The diagram is advantageous because it shows at a glance the relations among foods: which food has the higher lipid content, which has the more protein, etc. The triangle can be divided into regions. Sugars, syrups, and honey are almost pure carbohydrate, so they are represented by a small region near the origin of coordinates. Butter, margarine, and cooking oils are almost pure lipid and are represented near the top corner of the diagram. Most cheeses are in the neighborhood of $n^1 = n^2 = 0.5$ with very little carbohydrate. Therefore, they constitute a region near the middle of the right side of the triangle. Nuts lie in the upper middle part of the triangle, with pecans near the top and peanuts at the bottom of this region. Vegetables and grains form a region near the bottom of the diagram, with the common characteristic of low lipid content.

Combinations. Perhaps the greatest importance of the nutrition triangle is in determining the nutritional value of combinations of foods. *Any mixture of foods a and*

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[1] The term holor used in this paper was first used in 1963. Moon, P. & Spencer, D. E. (1963) A new mathematical representation of alternating currents. Tensor *14,* 110–121.

[2] Note that we are using superscripts instead of subscripts, in accordance with standard tensor notation for contravariant quantities.

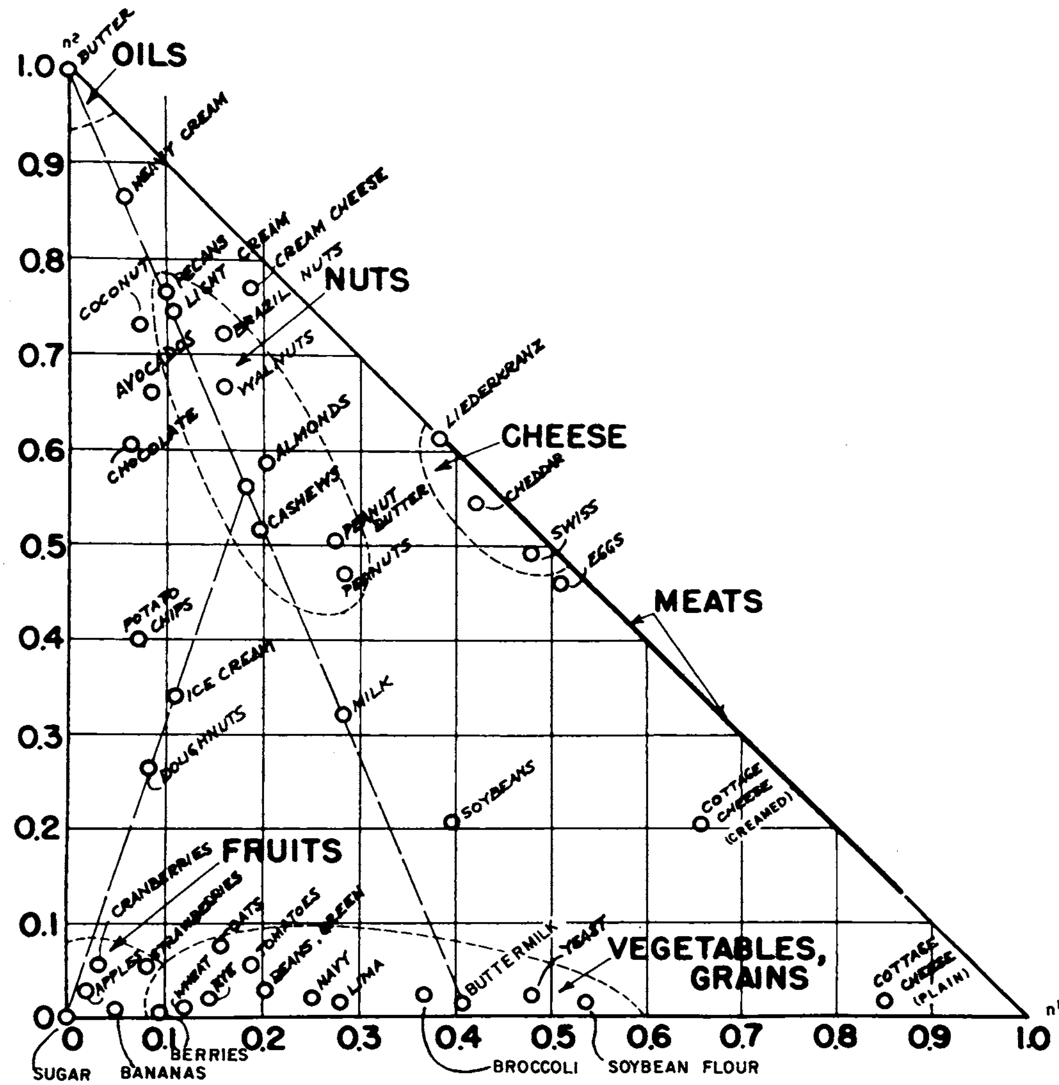

Fig. 1 The nutrition triangle. The horizontal scale refers to *protein*, the vertical scale refers to *lipid*.

b is represented in the diagram by a point on the straight line ab. In figure 1 for instance, different mixtures of buttermilk (0.402, 0.012) and butter (0.007, 0.989) are represented by the line between these terminal points. Reference to the triangle shows that milk, light cream, and heavy cream fall on this locus. As another example, consider ice cream (0.108, 0.337) obtained by mixing a cream with sugar or honey (0, 0). The line passing through these two points cuts the milk locus at (0.182, 0.560), indicating a cream between ordinary milk and light cream.

The *amounts* of the two ingredients can be calculated by the methods described in Mixtures in 2-space. For example, suppose that we mix milk and flour in various proportions, as represented by the line *ab* of figure 2. A scale of

$$x = \frac{\text{kg flour}}{\text{kg milk}}$$

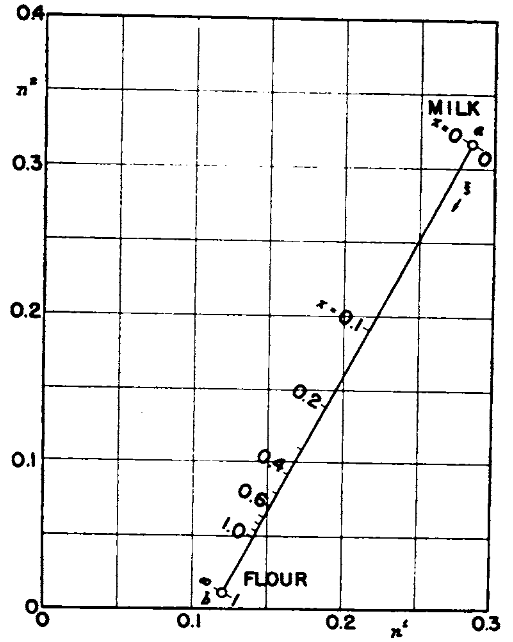

Fig. 2 Any mixture of two foods (a and b) is represented by a straight line segment in the nutrition triangle.

can be established as shown in the diagram. Thus, the exact proportions of flour and milk can be determined for any point on the locus.

Similarly, mixtures of *three* ingredients give all points within the shaded triangle, figure 3. This method can be extended to any number of ingredients, as suggested by figure 4. A convex polygon then includes all possible mixtures of the given constituents.

Such graphical construction shows immediately if a mixture of given ingredients can possibly result in a given proportion of nutrients. For example, suppose we take the point (0.100, 0.100) as our criterion. It is represented by the stars in figures 3 and 4. Because the shaded areas of figures 3 and 4 do not include the star, we can conclude immediately, without calculation, that *no possible mixture of the proposed ingredients can satisfy the nutritional specification.* By proper additions (in these cases, of sugar or fruit) however, the desired point can be reached. Proportions of the ingredients can then be determined by

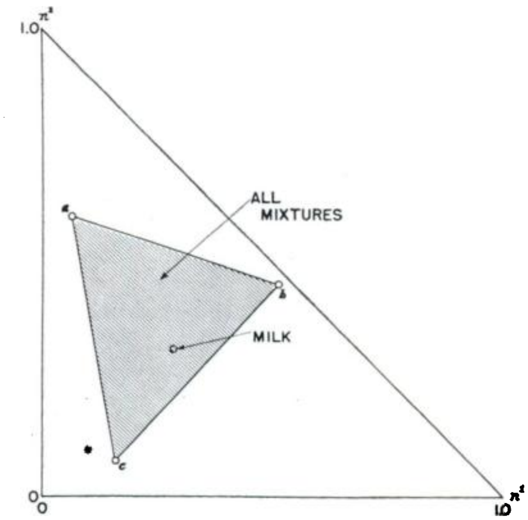

Fig. 3 A mixture of three foods (a, b, c) gives a point within the shaded area.

the method described in Mixtures in 2-space.

The nutrition holor. In the preceding sections, we have attempted to sketch some of the possibilities of nutritional geometry. In the remainder of the paper, we shall establish the mathematical basis for the method. Let us specify a given sample of food by three numbers repre-

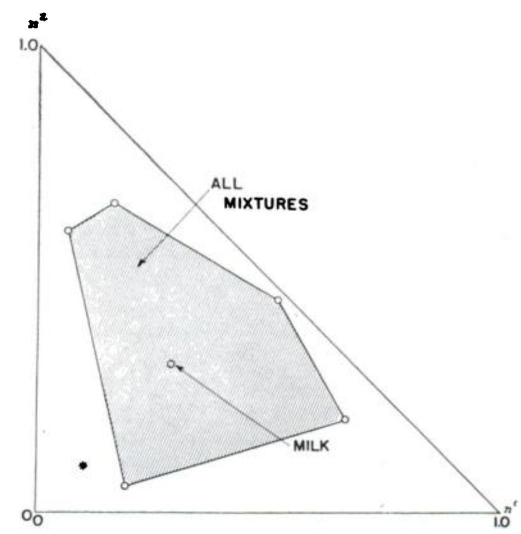

Fig. 4 A mixture of any number of foods is represented by a point within an area of the nutrition triangle.

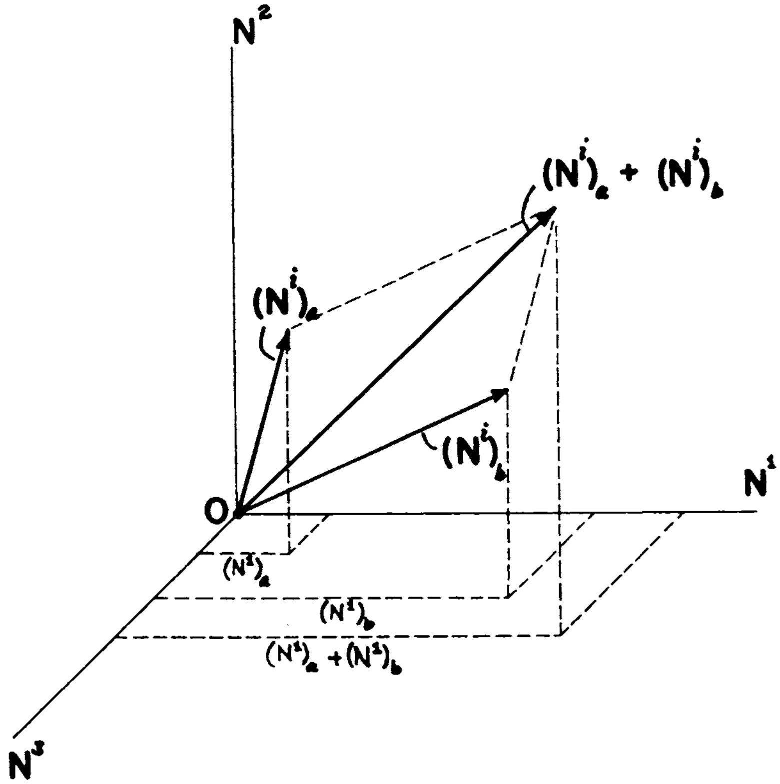

Fig. 5 Nutrition 3-space. Each food is represented geometrically by a point in this space.

senting kg of protein, kg of lipid, and kg of carbohydrate. Evidently, these numbers may be considered as "components" or *merates* (1) of a *nutrition holor* N^i:

kg protein

kg lipid

kg carbohydrate

$$N^i = (N^1, N^2, N^3). \quad \text{(Eq. 1)}$$

Equality of two holors $(N^i)_a$ and $(N^i)_b$ is obtained if and only if corresponding merates are equal:

$$(N^1)_a = (N^1)_b, \; (N^2)_a = (N^2)_b,$$

and

$$(N^3)_a = (N^3)_b.$$

Addition of two holors is obtained in the usual way by adding corresponding merates:

$$(N^i)_a + (N^i)_b = [(N^1)_a + (N^1)_b, (N^2)_a + (N^2)_b, (N^3)_a + (N^3)_b]. \quad \text{(Eq. 2)}$$

Addition is commutative and associative.

Multiplication by a scalar is meaningful:

$$kN^i = N^ik = (kN^1, kN^2, kN^3). \quad \text{(Eq. 3)}$$

For instance, if we have k identical samples of food, the total kg of protein is equal to k times the protein of each sample. Lipid and carbohydrate could be determined in a similar manner. Note, however, that *products of nutrition holors* have no physical significance. There is nothing corresponding to the familiar scalar and vector products of vector analysis (1).

Nutrition 3-space. Geometrically, N^i may be considered as a point in a nutritional 3-space (fig. 5). The axes represent pure protein, pure lipid, and pure carbohydrate, respectively. But for some purposes, it may be convenient to take other axes, representing milk, butter, and cheese, for instance. Because the system is a linear one, we can *transform* from one set of axes to another, using a *linear* or *affine* transformation (1):

$$N^{i'} = A_i^{i'} N^i \qquad \text{(Eq. 4)}$$

The transformation matrix is

$$A_i^{i'} = \begin{bmatrix} A_1^{1'} & A_2^{1'} & A_3^{1'} \\ A_1^{2'} & A_2^{2'} & A_3^{2'} \\ A_1^{3'} & A_2^{3'} & A_3^{3'} \end{bmatrix} \qquad \text{(Eq. 5)}$$

The holor N^i is an affine invariant that is independent of the coordinate system, though of course its merates $\{N^i\}$ depend on the coordinates in which it is expressed. Nutrition space is an *affine space* (2–5), not a Euclidean space. Holors N^i can be added, in accordance with Eq. 2, but magnitude $|N^i|$ cannot be obtained by the usual Euclidean formulas.

Addition of holors is valid in affine space. Suppose that we mix two food samples whose nutrition holors are $(N^i)_a$ and $(N^i)_b$. According to Eq. 2, the nutrition holor of the mixture is the "vector" sum shown in figure 5. Note that this property continues to hold if we transform coordinates by changing scales along the axes and rotating axes in an affine transformation.

Another property deals with multiplication by a scalar k. Because the same factor k appears in each merate, Eq. 3, a variation in k moves the point N^i along a straight line radiating from the origin. Thus, one may establish a family of lines through the origin, each line representing a different kind of food. For each kind of food, the origin represents *zero* amount of that food, with amounts increasing linearly as we move outward from *0*. Instead of thinking in terms of merates, therefore, we may characterize a given sample of food by *magnitude* along a line and *direction* of this line from the axes. As noted previously, however, magnitude is not the sum of the squares of the merates.

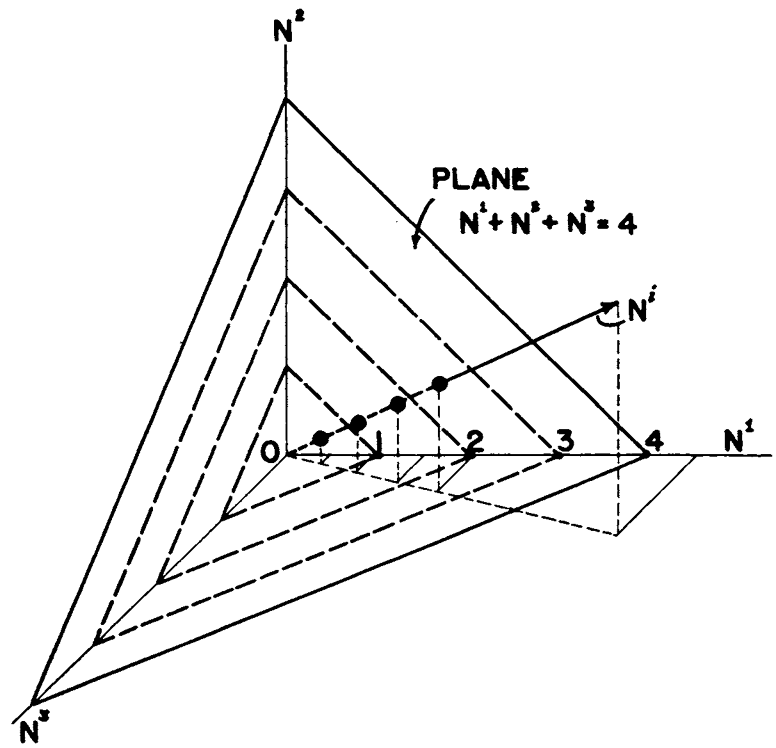

Fig. 6 Planes of constant magnitude in nutrition 3-space.

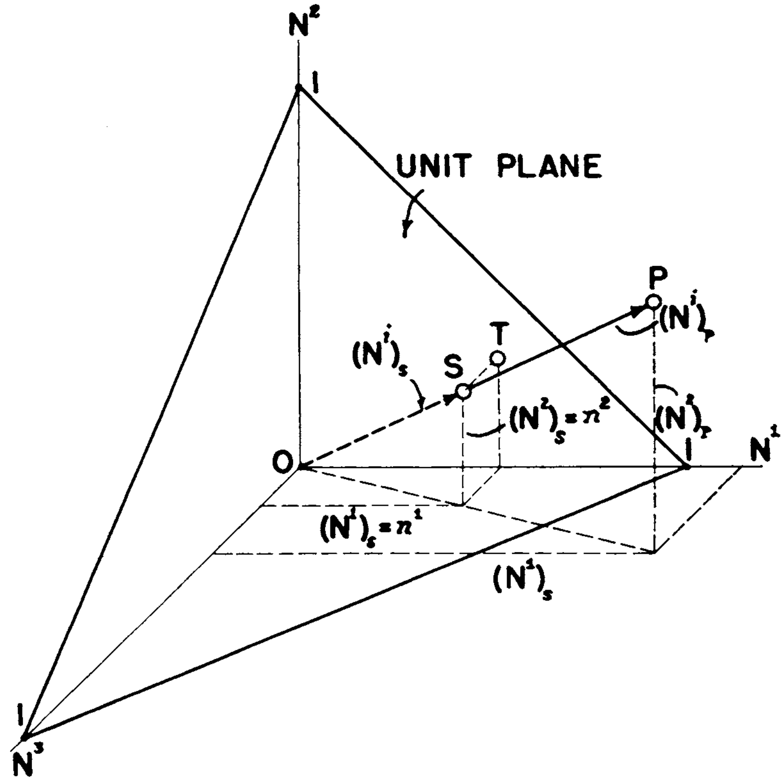

Fig. 7 Relation between nutrition 3-space and the nutrition triangle.

Magnitude. A method of separating these two quantities consists in introducing planes into nutrition space. The equation of a plane may be written

$$AN^1 + BN^2 + CN^3 = D. \quad \text{(Eq. 6)}$$

For the special case of equal intersections of the plane and the three axes,

$$N^1 + N^2 + N^3 = D. \quad \text{(Eq. 6a)}$$

A *magnitude* scale can now be introduced by arbitrarily defining the magnitude of any point in the plane, Eq. 6a, as D (fig. 6). This convention establishes a linear (but different) scale of magnitude on each line radiating from the origin. *Magnitude is the total mass (kg) of the nutrients* ($N^1 + N^2 + N^3$) *in a given sample of food.*

Energy. An alternative way of obtaining a meaning for the word "magnitude" would be to use the convention for *energy,*

$$E = 4{,}000\, N^1 + 9{,}000\, N^2 + 4{,}000\, N^3. \quad \text{(Eq. 7)}$$

Here $\{N^i\}$ is in kg as usual, and E is in kcal. This expression is of the form of Eq. 6 and therefore represents a plane. But the plane does not have equal intersections on the three axes.

Eq. 7 is important in its own right, because it allows *calories* to be obtained from any nutrition holor N^i. We do not feel, however, that it should replace our previous definition of *magnitude.*

Nutrition 2-space. We now return to the specification of *character.* The simplest definition employs the unit plane ($D = 1$) of figure 6. This plane (fig. 7) has the equation

$$N^1 + N^2 + N^3 = 1. \quad \text{(Eq. 6b)}$$

Each point from the origin pierces the plane at a definite point S; so each point on the slanting plane represents a definite *character*, divorced from magnitude.

It is now desirable to express the coordinates of point S in terms of the coordinates of point P (fig. 7) of a definite food sample. Evidently,

$$\left.\begin{aligned}(N^1)_S &= \frac{(N^1)_P}{(N^1+N^2+N^3)_P},\\ (N^2)_S &= \frac{(N^2)_P}{(N^1+N^2+N^3)_P},\\ (N^3)_S &= \frac{(N^3)_P}{(N^1+N^2+N^3)_P}.\end{aligned}\right\} \quad \text{(Eq. 8)}$$

Because of Eq. 6b, only two of these three quantities are independent. It is convenient to simplify the notation and write n^1 instead of $(N^1)_s$ and n^2 instead of (N^2). Point S can then be projected into T of the N^1N^2 plane to give the right triangle of figure 1, and

$$\left.\begin{aligned}n^1 &= \frac{N^1}{N^1+N^2+N^3},\\ n^2 &= \frac{N^2}{N^1+N^2+N^3},\end{aligned}\right\} \quad \text{(Eq. 9)}$$

where the capital letters represent merates of point P.

Mixtures in 2-space. Nutritional values of *combinations* of foods are usually obtained by adding the N^i-holors in 3-space in accordance with Eq. 2. It would be helpful, however, if we could develop an equation by means of which the position of a given mixture could be evaluated directly from the 2-space without reverting to the 3-space.

Consider two samples of food a and b (fig. 2) specified by masses $(m)_a$ and $(m)_b$ and by character holors $(n^i)_a$ and $(n^i)_b$. As mentioned previously, all mixtures of foods a and b lie on the line ab of the nutrition triangle. For 100% of food a, we have point a of the triangle; for 100% of food b, we have point b. For mixtures, point c shifts along the line ab, and its exact position can be calculated by "vector" addition in the 3-space. Let us establish a scale ξ running from 0 to 1 along the line ab:

$$\xi = \frac{(n^1)_c - (n^1)_a}{(n^1)_b - (n^1)_a}. \quad \text{(Eq. 10)}$$

For the special case of ab parallel to the n^2 axis, Eq. 10 reduces to 0/0. We must then change all the 1 superscripts to 2 in Eqs. 10 to 15.

But

$$(N^i)_c = (N^i)_a + (N^i)_b, \quad \text{(Eq. 11)}$$

and

$$\left.\begin{aligned}(N^i)_a &= (B^i)_a(m)_a,\\ (N^i)_b &= (B^i)_b(m)_b,\end{aligned}\right\} \quad \text{(Eq. 12)}$$

where $(B^i)_a$ and $(B^i)_b$ are obtained from the food tables. Also,

$$\left.\begin{aligned}(n^1)_a &= \frac{(N^1)_a}{(N^1+N^2+N^3)_a},\\ (n^1)_b &= \frac{(N^1)_b}{(N^1+N^2+N^3)_b},\\ (n^1)_c &= \frac{(N^1)_a+(N^1)_b}{(N^1+N^2+N^3)_c},\end{aligned}\right\} \quad \text{(Eq. 13)}$$

and

$$(N^1+N^2+N^3)_a = [(B^1)_a + (B^2)_a + (B^3)_a](m)_a = (B)_a(m)_a,$$

where B denotes the scalar sum of the tabulated merates of B_j^i for a given j.

Similarly,

$$(N^1+N^2+N^3)_c = (B)_a(m)_a + (B)_b(m)_b. \quad \text{(Eq. 14)}$$

Thus,

$$\left.\begin{aligned}(n^1)_a &= \frac{(B^1)_a}{(B)_a}, \quad (n^1)_b = \frac{(B^1)_b}{(B)_b},\\ (n^1)_c &= \frac{(B^1)_a(m)_a + (B^1)_b(m)_b}{(B)_a(m)_a + (B)_b(m)_b}.\end{aligned}\right\} \quad \text{(Eq. 15)}$$

Substitution into Eq. 10 gives

$$\xi = \frac{zx}{zx+1}, \quad \text{(Eq. 16)}$$

where

$$z = (B)_b/(B)_a \quad \text{and} \quad x = (m)_b/(m)_a.$$

If the coordinates of point c are desired, they can be obtained from Eq. 17:

$$\left.\begin{aligned}(n^1)_c &= \frac{(B^1)_a + (B^1)_b x}{(B)_a + (B)_b x},\\ (n^2)_c &= \frac{(B^2)_a + (B^2)_b x}{(B)_a + (B)_b x}.\end{aligned}\right\} \quad \text{(Eq. 17)}$$

For mixtures of *three* ingredients (a, b, c), we obtain all points within the resulting triangle (fig. 3). Evidently, the result d of any given mixture is obtained by simple extension of Eq. 17:

$$\left.\begin{aligned}(n^1)_d &= \frac{(B^1)_a + (B^1)_b x + (B^1)_c y}{(B)_a + (B)_b x + (B)_c y},\\ (n^2)_d &= \frac{(B^2)_a + (B^2)_b x + (B^2)_c y}{(B)_a + (B)_b x + (B)_c y},\end{aligned}\right\} \qquad \text{(Eq. 18)}$$

where

$$x = (m)_b/(m)_a, \; y = (m)_c/(m)_a.$$

The extension to any number of ingredients is obvious.

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